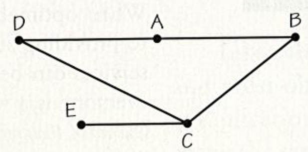
which links (called edges) are used to connect vertices, denoting that the connected vertices have a certain relationship. (p. 4)
Management science A discipline in which mathematical methods are applied to management problems in pursuit of optimal solutions that cannot readily be obtained by common sense. (p. 1)
Operations research (OR) Another name for management science. (p. 1)
Optimal solution When a problem has various solutions that can be ranked in preference order (perhaps according to some numerical measure of "goodness"), the optimal solution is the best-ranking solution. (p. 3)
Path A connected sequence of edges in a graph. (p. 4)
Valence (of a vertex) The number of edges touching that vertex. (p. 7)
Vertex A point in a graph where one or more edges end. (p. 4)

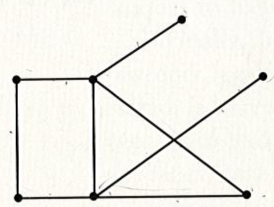
SKILLS CHECK

1. What is the valence of vertex *A* in the graph below?

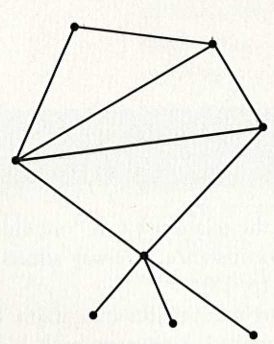


(a) 2
(b) 1
(c) 3

2. The number of vertices in the graph below is _____ , while the number of edges in this graph is ________ .

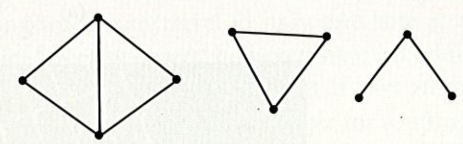


3. The valences of the vertices in the accompanying graph listed in non-increasing order are

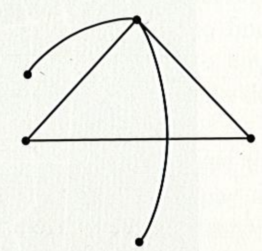


(a) 5, 4, 3, 3, 2, 1, 1, 1.
(b) 1, 3, 4, 4, 5, 5.
(c) 5, 5, 4, 3, 3, 1

4. The graph shown below is not connected because it consists of ________ parts.

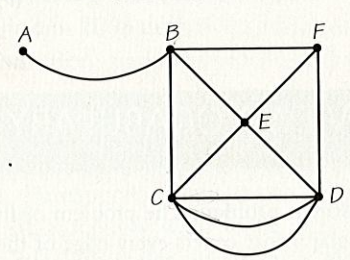


5. The graph below has



(a) four vertices and six edges.
(b) four vertices and four edges.
(c) five vertices and five edges.

6. The alphabetically ordered list of even-valent vertices of the graph below is ______ , ______ .



7. Which of the following statements is true about a *path?*
(a) A path always forms a circuit.
(b) A path is always connected.
(c) A path can visit any vertex only once.

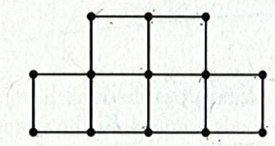
8. If a graph consists of four vertices and every pair of vertices is connected by a single edge, the number of edges in the graph is exactly _______ .

9. It is not possible for a graph to have five vertices of valence 3 and six vertices of valence 4 because
(a) there are no graphs with exactly 11 vertices.
(b) a graph cannot have an even number of 4-valent vertices.
(c) a graph cannot have an odd number of odd-valent vertices.

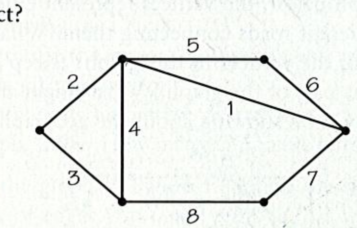
10. If a graph is connected and has seven vertices, the graph must have at least _________ edges.

11. For which of the situations below is it most desirable to find an Euler circuit or an efficient eulerization of the graph?
(a) Sweeping the sidewalks of a small town
(b) Planning a new highway
(c) Planning a parade route in Muncie, Indiana

12. The minimum number of edges which must be duplicated to create a best possible eulerization of the following graph is ______ .

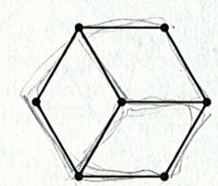


13. Consider the path represented by the sequence of numbered edges on the graph below. Which statement is correct?



(a) The sequence of numbered edges forms an Euler circuit.
(b) The sequence of numbered edges traverses each edge exactly once but is not an Euler circuit.
(c) The sequence of numbered edges forms a circuit but not an Euler circuit.

14. For the graph below, the minimum *total* number of edges which constitutes a tour of the graph, starting and ending at the same vertex, and which visits each edge at least once, is _______ .



15. Suppose each vertex of a graph represents a baseball team and each edge represents a game played by two baseball teams. If the resulting graph is not connected, which of the following statements must be true?
(a) At least one pair of teams never played a game.
(b) At least one team played every other team.
(c) The teams play in distinct leagues.

16. If a graph has six vertices of odd valence, the absolute minimum number of edges that must be added (duplicated) to eulerize the graph is ________ .

17. Suppose the edges of a graph represent streets that must be plowed after a snowstorm. To eulerize the graph, four edges must be added. The real-world interpretation of this is that
(a) four streets will not be plowed.
(b) four streets will be traversed twice.
(c) four new streets would be built.

18. For each of the following situations, decide whether a graph or a digraph seems a more reasonable model:
(a) A system of hiking trails: ______________ .
(b) An electrical wiring plan for a home: __________ .
(c) A bus route map: ______________________ .

19. Suppose a civic club offers several craft courses, and each club member can choose to participate in up to two different courses. Let each vertex of a graph represent one of these courses and each edge represent a club member who wants to take the two courses represented by the vertices at its endpoints. What can be said about the vertices in the resulting graph whose valence is zero?
(a) There are no vertices whose valence is zero.
(b) These vertices represent courses that can occur at the same time without displeasing any club member.
(c) These vertices represent the least popular courses.

20. If the valences of the vertices of a graph *G* are: 5, 4, 4, 4, 3, 2, 2, and 2, the number of vertices of *G* is _______ and the number of edges of *G* is _______ .

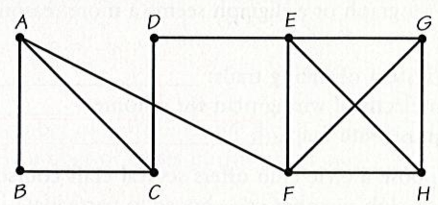
CHAPTER 1 EXERCISES

■ Challenge ◆ Discussion

1.1 Euler Circuits
1.2 Finding Euler Circuits

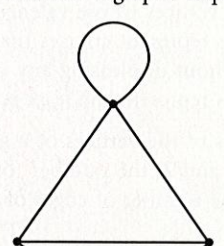
1. In the graph below, the vertices represent houses and two vertices are joined by an edge if it is possible to drive between the two houses in under 10 minutes.

(a) How many vertices does the graph have?
(b) How many edges does the graph have?
(c) What are the valences of the vertices in this graph?
(d) Based on the information given by the graph below, for which houses, if any, is it possible to drive to all the other houses in less than 20 minutes?
(e) Based on the graph below, from house *B* which houses require a trip of longer than 20 minutes?

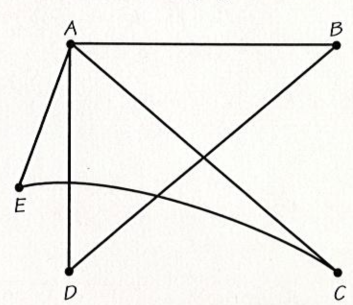


2. (a) Redraw the graph in Figure 1.2 to obtain a graph which has the same information where the edges only meet other edges at vertices.
(b) List all the routes that start on the U.S. side of the Atlantic Ocean and cross the ocean once and immediately.

3. (a) Is the figure below a graph? Explain your answer.

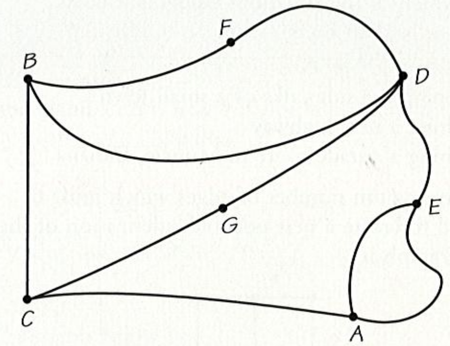


(b) The graph below has edges that "cross" at points that are not vertices of the graph. Which edges are these?



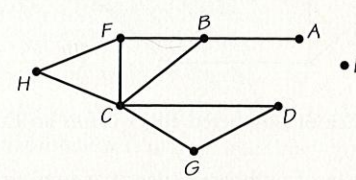
(c) How many vertices and edges are there in the preceding graph?

4. The graph below shows the stores and roads connecting them in a small shopping mall.

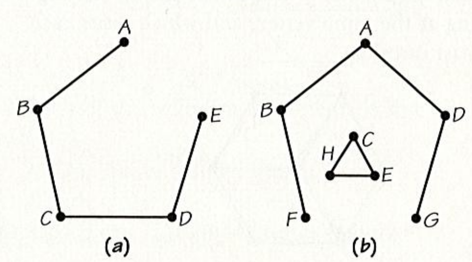


(a) How many stores does the mall have?
(b) How many roads connect up the stores in the mall?
(c) Write down a path from *C* to *F*.
(d) Write down a path from *E* to *B*.

5. In the graph below, the vertices represent cities and the edges represent roads connecting them. What are the valences of the vertices in this graph? (Keep in mind that *E* is part of the graph.) What might the valence of city *E* be showing about the geography?



6. In the two graphs below, the vertices represent cities and the edges represent roads connecting them. In which graphs could a person located in city *A* choose any other city and then find a sequence of roads to get from *A* to that other city?



7. Refer to the figure in Exercise 4.
(a) Write down a circuit that includes the vertices *C* and *D* but does not start or end at either of these vertices.
(b) If two paths are considered different if they use different edges, write down:
- **(i)** two different paths from *B* to *D*.
- **(ii)** three different paths from *C* to *F*.
- **(iii)** a circuit that has four edges.

8. Jack and Jill are located in Miami and want to fly to Berlin (see Figure 1.2).
(a) Find three paths for them to carry out this trip.
(b) What is the largest number of paths that can be used to carry out this trip that do not repeat a vertex (city)?
(c) Explain why it is reasonable not to want to repeat a vertex in this situation.

9. (a) How many vertices and edges does the graph in Figure 1.6 have?
(b) How many vertices and edges does the graph in Figure 1.7 have?
(c) How many vertices and edges does the graph in Figure 1.8a have?

10. (a) Add up the numbers you get for the valences of the vertices in Figure 1.6.
(b) Add up the numbers you get for the valences of the vertices in Figure 1.7.
(c) Add up the numbers you get for the valences of the vertices in Figure 1.8a.
(d) Describe the pattern you see in the answers you got for parts (a) through (c).
(e) Show that the pattern describes a fact that is true for any graph. (*Hint:* How many endpoints does an edge have?)

11. In the graph in Figure 1.8a, find the smallest possible number of edges you could remove that would disconnect the graph.

12. In the graphs in Figure 1.17, find the smallest possible number of edges you could remove that would disconnect the graph.

13. Draw a graph with eight vertices that is connected where
(a) each vertex has valence 3.
(b) each vertex has valence 4.
(c) Do all graphs with eight vertices having valence 2 have the same number of edges?

14. Is it possible that a street network gives rise to a disconnected graph? If so, draw such a network of blocks and streets and parking meters (in the style of Figure 1.12a). Then draw the disconnected graph it gives rise to.

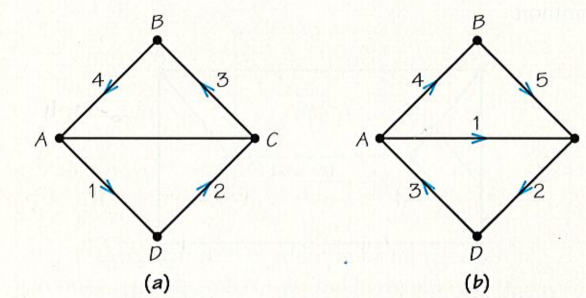
15. (a) Draw a connected graph with six vertices, all of whose vertices have valence 2.
(b) Draw a disconnected graph with 6 vertices, all of whose vertices have valence 2.

16. (a) Draw a graph where every vertex has valence of at least 3 but where removing a single edge disconnects the graph.
(b) In what urban settings might a road network be represented by a graph that has an edge whose removal would disconnect the graph?

17. (a) Find a graph where the valences of the seven vertices of the graph are 1, 2, 2, 3, 3, 3, 4.
(b) Find another graph with the same valences as above that is "different" from the one you found for part (a).

18. For some services provided along streets, it may matter whether the roads are one-way or two-way. Give some examples where the street directions do and do not matter for our graph model analysis.

◆ **19.** A postal worker is supposed to deliver mail on all streets represented by edges in the graph below by traversing each edge exactly once. The first day the worker traverses the numbered edges in the order shown in (a), but the supervisor is not satisfied–why? The second day the worker follows the path indicated in (b), and the worker is unhappy–why? Is the original job description realistic? Why?

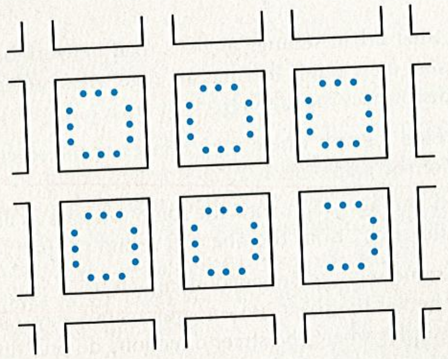


20. For the street network in Exercise 19, draw the graph that would be useful for routing a snowplow. Assume that all streets are two-way, one lane in each direction, and that you need to pass down each lane separately.

21. Find an efficient route for the snowplow to follow in the graph you drew in Exercise 20.

22. (a) Give examples of services that could be performed by a vehicle that moved in the direction of traffic down either lane of a two-way street.
(b) Give examples of services that would probably require a vehicle to travel down each of the lanes of a two-way street (in the direction of traffic for that lane) to perform the service.

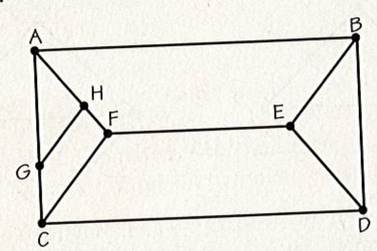
23. For the street network shown below, draw the graph that would be useful for finding an efficient route for checking parking meters. (*Hint:* Notice that not every sidewalk has a meter; see Figure 1.12.)



24. (a) For the street network in Exercise 23, draw a graph that would be useful for routing a garbage truck. Assume that all streets are two-way and that passing once down a street suffices to collect from both sides.

(b) Do the same problem on the assumption that one pass down the street suffices to collect from only one side.

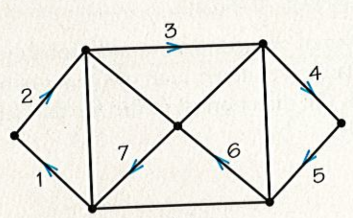
25. (a) In the graph below, find the largest number of paths from A to F that do not have any edges in common.



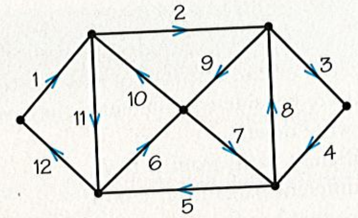
(b) Verify that the largest number of paths with no edges in common between any pair of vertices in this graph is the same.

(c) Why might one want to be able to design graphs such that one can move between two vertices of the graph using paths that have no edges in common?

26. Examine the paths represented by the numbered sequences of edges in both parts of the figure below. Determine whether each path is a circuit. If it is a circuit, determine if it is an Euler circuit.



(*a*)



(*b*)

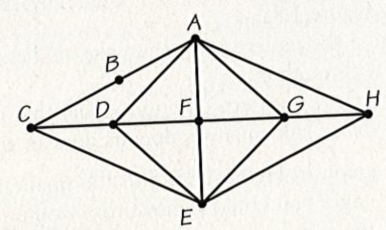
27. In Figure 1.13c, suppose we started an Euler circuit using this sequence of edges: 6, 7, 8, 9 (ignore existing arrows on the edges). What does our guideline for finding Euler circuits tell you *not* to do next?

28. In Figure 1.8b, suppose we started an Euler circuit using this sequence of edges: 14, 13, 8, 1, 4 (ignore existing arrows on the edges). What does our guideline for finding Euler circuits tell you *not* to do next?

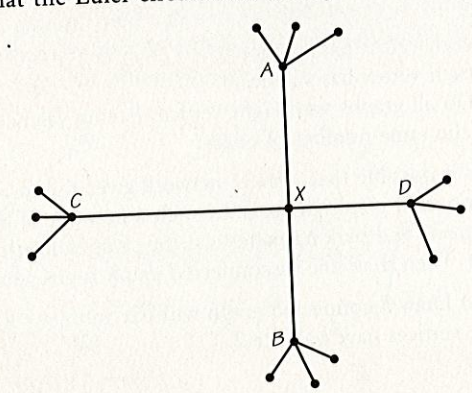
29. Find an Euler circuit on the graph of Figure 1.15c (including the blue edges).

30. Find Euler circuits in the right-hand graphs in Figures 1.17a and 1.17b.

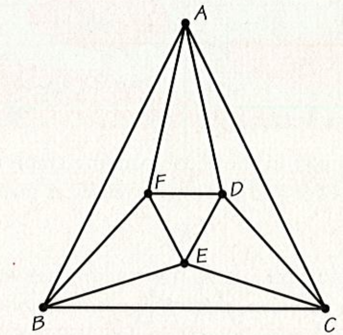
31. In the following graph, we see a territory for a parking-control officer that has no Euler circuit. How many sidewalks (edges) need to be omitted in order to enable us to find an Euler circuit? What effect would this have in the associated real-world situation?



32. An Euler circuit visits a four-valent vertex X, such as the one in the accompanying graph, by using the edges AX and XB consecutively, and then using CX and XD consecutively. When this happens, we say that the Euler circuit cuts through at X.



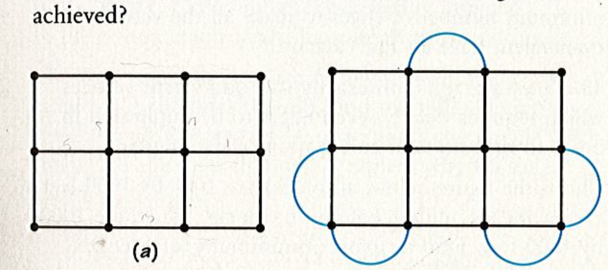
Suppose G is a four-valent graph such as that in the diagram below. Is it possible to find an Euler circuit of this graph that never cuts through any vertex? Explain why it might be desirable to find an Euler circuit of this special kind in an applied situation.



1.3 Beyond Euler Circuits

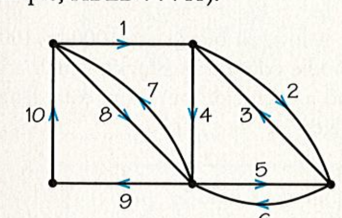
1.4 Urban Graph Traversal Problems

33. Find an Euler circuit on the eulerized graph (b) of the following figure. Use it to find a circuit on the original graph (a) that covers all edges and reuses edges only five times. Can fewer than five reused edges be achieved?

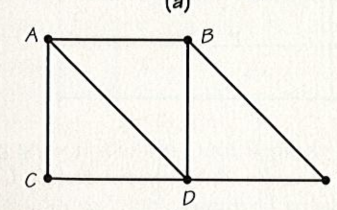


(*a*) (*b*)

34. Squeeze the circuit shown in graph (a) below onto graph (b). Show your answers by writing numbered arrows on the edges and by listing a sequence of vertices (for example, *ABEB . . . A*).

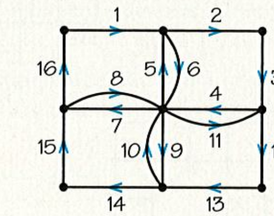


(*a*)

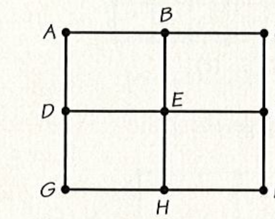


(*b*)

Then squeeze the circuit shown in graph (c) onto graph (d). Show your answers by writing numbered arrows on the edges and by listing a sequence of vertices.

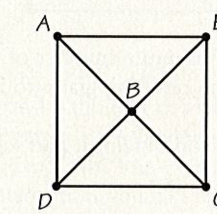


(*c*)

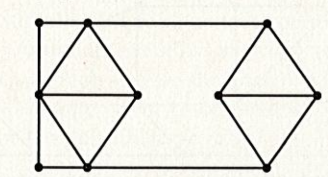


(*d*)

35. A college campus has a central square with sides arranged as shown by the edges in the graph below. Show how all these sidewalks can be traversed at least once in a tour that starts and ends at the same vertex.



36. In the graph below, add one or more edges to produce a graph that has an Euler circuit.



37. Eulerize these rectangular street networks using the same patterns that would be used by the edge walker described in the text.

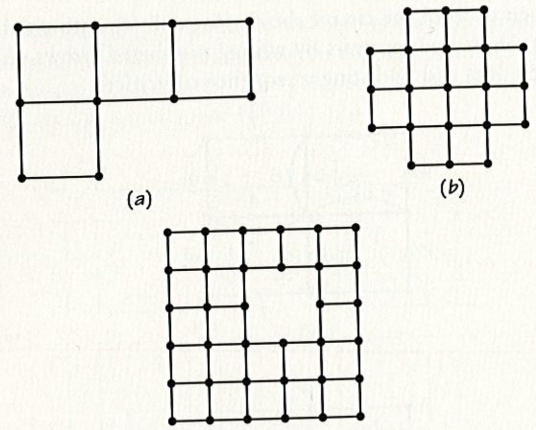
(a) A 5 × 5 rectangle

(b) A 4 × 5 rectangle

(c) A 6 × 6 rectangle

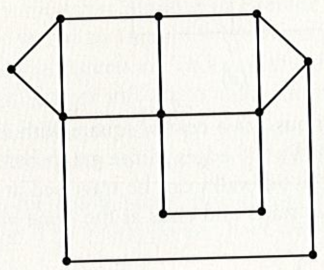
(d) Can you find an eulerization with nine added edges for a 2-by-7-block rectangular street network? Can you do better than nine added edges?

38. Find good eulerizations for the following graphs, using as few duplicated edges as you can. See "Finding Good Eulerizations" for hints.



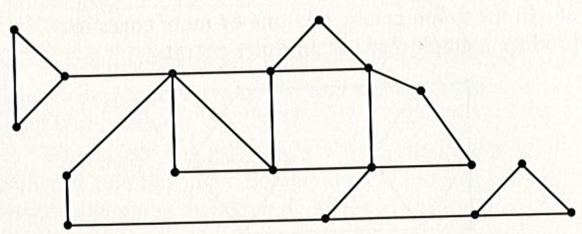
(a) (b) (c)

39. For the following graph:



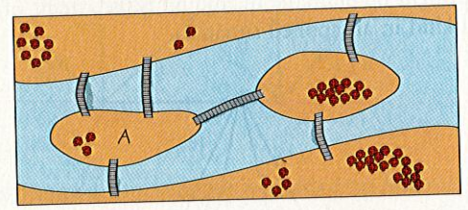
(a) Determine the minimum number of edges that have to be removed for the resulting graph to have all even-valent vertices.
(b) Does the graph you obtain in part (a) have an Euler circuit?

For the graph below:

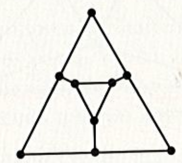


(c) Determine the minimum number of edges that have to be removed for the resulting graph to have all even-valent vertices.
(d) Does the graph you obtain in part (c) have an Euler circuit?

40. The following figure shows a river, some islands, and bridges connecting the islands and riverbanks. A charity is sponsoring a race in which entrants have to start at *A*, go over each bridge at least once, and end at *A*. Draw a graph that would be useful for finding a route that requires the least recrossing of bridges. Show what that route would be. (*Historical note:* This situation resembles the one that inspired Leonhard Euler's 1736 "recreational mathematics" problem that resulted in the first work in graph theory.)



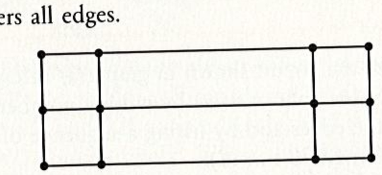
41. Find a circuit in the accompanying graph that covers every edge and has as few reuses as possible.



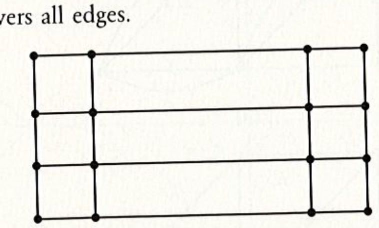
42. (a) Discuss the difference between the problem of:
(i) Adding the minimum number of edges to a graph to make all its vertices even-valent, and
(ii) Finding the best eulerization of a connected graph.
(b) In (i) must the graph that results from adding a minimum number of edges to make all the vertices even-valent have an Euler circuit?

43. Draw a graph with exactly two odd-valent vertices which requires exactly seven edges to be duplicated in order to find the best eulerization of the graph.

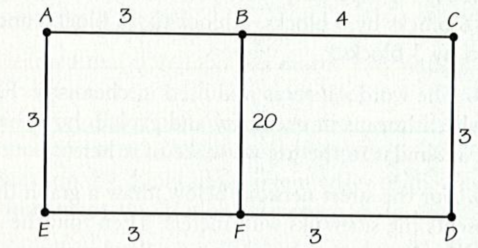
44. In the figure below, all blocks are 1000-by-1000 feet, except for the middle column of blocks, which are 1000-by-4000 feet. Find a circuit of minimum total length that covers all edges.



45. In the figure below, all blocks are 1000-by-1000 feet, except for the middle column of blocks, which are 1000-by-4000 feet. Find a circuit of minimum total length that covers all edges.

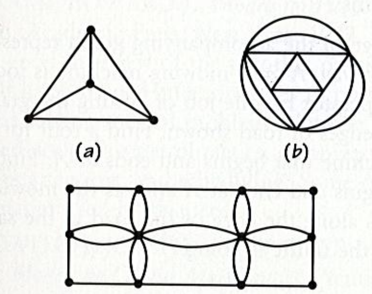


46. (a) Find the cheapest route in the following graph, where one starts at vertex *A*, finishes at vertex *A*, and traverses each edge at least once. The cost of a route is computed by summing the numbers along the edges that one uses.
(b) How many edges are repeated in the minimal-cost route?



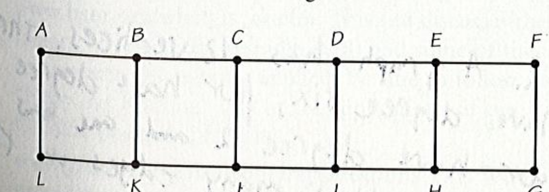
(c) Discuss the implications of this example for the relation between finding good eulerizations of graphs and the problem of finding cheap routes that start and end at the same vertex and traverse each edge at least once.
(d) The physical edge with cost 20 in the diagram is not physically longer than other edges with lower costs attached to them. Explain why in an urban setting it might make sense to assign two stretches of street of similar length very different "costs" for traversing them.
(e) What are some different meanings that "weights" (for example, traffic volume) potentially assigned to edges in a graph might have in an urban setting?

47. Which graphs (see figures below) have Euler circuits? In the ones that do, find the Euler circuits by numbering the edges in the order the Euler circuit uses them. For the ones that don't, explain why no Euler circuit is possible.



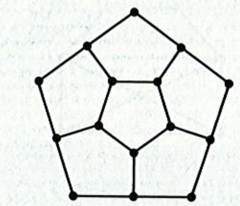
(a) (b) (c)

48. Eulerize the graph below by using four new edges. Find an Euler circuit in the eulerized graph and use that circuit to find a circuit of the original graph that covers all edges but reuses edges only four times. How many different ways can the four edges be chosen?

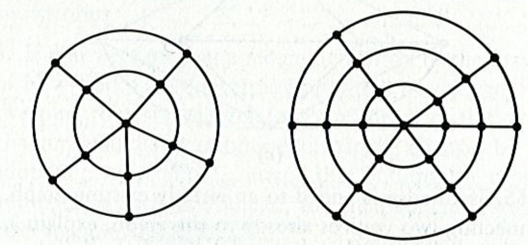


■ **49.** A graph *G* represents a street network to be traveled by a postal worker who must traverse every street twice, once for each side of the street. In graph *G*, the edges represent sidewalks. Does such a graph always have an Euler circuit? Explain your answer.

50. In the graph below, find a circuit that covers every edge and has as few reuses as possible.



51. (a) Find the best eulerizations you can for the two graphs below.



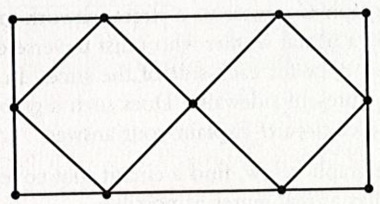
(a) (b)

(b) Graph (a) can be thought of as having five rays and two circles, and graph (b) as having six rays and three circles. Draw a graph with four rays and four circles and find the best eulerization you can for this graph.
(c) Find a "formula" involving *r* and *s* for the smallest number of edges needed to eulerize a graph of this type having *r* rays and *s* circles.

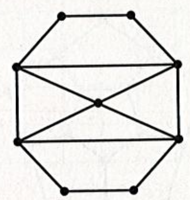
■ **52.** Suppose that for a certain connected graph it is possible to disconnect it by removing one edge. Explain why such a graph (before the edge is removed) must have at least one vertex of odd valence. (*Hint:* Show that it cannot have an Euler circuit.)

53. Can you draw a graph with six vertices where the valence of each vertex is 5?

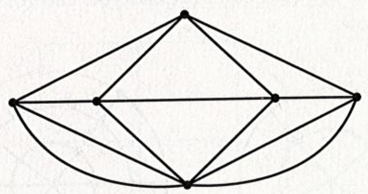
54. Each of the following graphs represent the sidewalks to be cleaned in a fancy garden (one pass over a sidewalk will clean it). Can the cleaning be done using an Euler circuit? If so, show the circuit in each graph by numbering the edges in the order the Euler circuit uses them. If not, explain why no Euler circuit is possible.



(a)



(b)

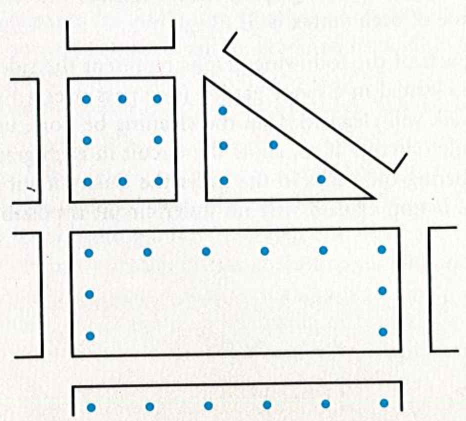


(c)

■ **55.** If an edge is added to an already existing graph, connecting two vertices already in the graph, explain why the number of vertices with odd valence has the same parity before and after. (This means if it was even before, it is even after, while if it was odd before, it remains odd.)

■ **56.** Any graph can be built in the following fashion: Put down dots for the vertices, then add edges connecting the dots as needed. When you have put down the dots, and before any edges have been added, is the number of vertices with odd valence an even number or an odd number? What is the number of vertices with odd valence when all the edges have been added (see Exercise 55)?

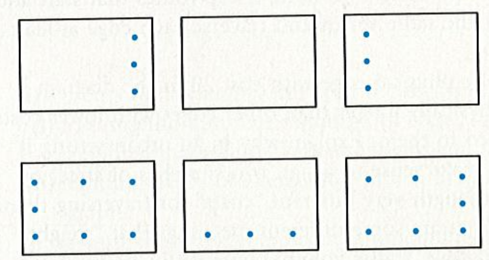
57. Draw the graph for the parking-control territory shown in the figure below. Label each vertex with its valence and determine if the graph is connected.



58. If a rectangular street network is r blocks by s blocks, find a formula for the minimum number of edges that must be added to eulerize a graph representing the network in terms of r and s. (*Hint:* Treat the case $r = 1$ separately. Test your formula with the cases 6 blocks by 5 blocks, 6 blocks by 6 blocks, and 5 blocks by 3 blocks.)

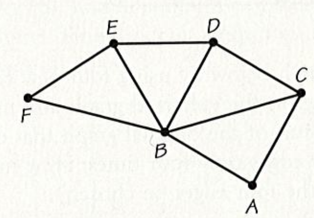
◆ **59.** The word *valence* is also used in chemistry. Find out what it means in chemistry and explain how this usage is similar to the use we make of it here.

■ **60.** For the street network below, draw a graph that represents the sidewalks with meters. Then find the minimum-length circuit that covers all sidewalks with meters. If you drew the graph as we recommended, you would find that the shortest circuit has length 18 (it reuses every edge).



But the meter checker comes to you and says: "I don't know anything about your theories, but I have found a way to cover the sidewalks with meters using a circuit of length 10. My trick is that I don't rule out walking on sidewalks with no meters." Explain what he means and discuss whether his strategy can be used in other problems.

61. Each edge of the accompanying graph represents a two-lane highway. A grass-mowing machine is located at *A*, and its operator has the job of cutting the grass along each of the edges of road shown. Find a tour for the mowing machine that begins and ends at *A*. Find such a tour that begins and ends at *A* and, as the mowing is done, moves along the edge of the road in the same direction as the traffic is going.



62. A graph has 12 vertices. three have degree six, four have degree four, four have degree 2 and one has degree 2. How many edges in the graph?

APPLET EXERCISES

To do these exercises, go to www.whfreeman.com/fapp8e.

Eulerizing a Graph

We learned that if a graph has exactly two vertices with odd valences, then an Euler circuit does not exist—but an Euler path does. It is also possible to produce an Euler circuit through the process of eulerization, by duplicating certain edges of the graph. But how many duplications are necessary to obtain an Euler circuit? Investigate this problem and more general related topics using the *Eulerizing a Graph* applet.

Euler Circuits

We know that if all the vertices have even valence, then an Euler circuit exists. Try your hand at finding such circuits in the *Euler Circuit* applet.

WRITING PROJECTS

1. Write a memo to your local department of parking control (or police department) in which you suggest that management science techniques like the ones in this chapter be used to plan routes. Assume that the person to whom you are writing is not extensively trained in mathematics but is willing to read through some technical material, provided you make it seem worth the trouble.

2. Do the same as in Writing Project 1, but to the department in charge of spreading salt on roads after snowstorms.

3. If you were making a recommendation to the mayor of New York City concerning proposed new street-sweeping routes, designed using the theory of this chapter, would you recommend that the changes be adopted or not? Write a memo that outlines the pros and cons as fairly as you can, and then conclude with your recommendation.

SUGGESTED READINGS

BELTRAMI, EDWARD J., *Models for Public Systems Analysis,* Academic Press, New York, 1977. This book gives a good overview of the way that operations research has provided and continues to provide new tools for solving societal problems. Among the ideas discussed are police patrol tactics, organization of emergency services, and scheduling. Some of the mathematics used is advanced.

MALKEVITCH, JOSEPH, and WALTER MEYER, *Graphs, Models and Finite Mathematics,* Prentice-Hall, Englewood Cliffs, NJ, 1974. This introductory book includes much of the same material as presented here but provides more details of the proofs and uses somewhat different algorithms for solving the problems involved.

The following books treat many of the topics discussed here as well as shortest-path problems and matching problems, and they formulate some problems in more realistic terms:

ROBERTS, FRED S., AND BARRY TESMAN, *Applied Combinatorics,* Second Edition, Pearson Prentice Hall, Upper Saddle River, NJ, 2004.

TUCKER, ALAN. *Applied Combinatorics,* Third Edition, Wiley, New York, 1995.

SUGGESTED WEB SITES

www.hsor.org/what_is_or.cfm This site discusses the history of operations research (OR) and some of the areas where OR is being applied. Be sure to follow the "Networks Routing" link to see applications of the Chinese postman problem.

www.geom.uiuc.edu/~doty/applications This Web page provides some examples of how to apply Euler circuits.

www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Euler.html This essay discusses the numerous contributions that Euler made to mathematics, and provides biographical information about him.

www.ams.org/featurecolumn/archive/urban-geom.html This Web page includes an introduction to how graph theory has provided tools for urban operations research.