The critical-path method was popularized and came into wider use as a consequence of the *Apollo* project. This project, which aimed at landing a man on the moon within 10 years of 1960, was one of the most sophisticated projects in planning and scheduling ever attempted. The dramatic success of the project can be attributed partly to the use of critical-path ideas and the related program evaluation and review technique (PERT), which helped keep the project on schedule.

In Chapter 3, we will see how mathematical ideas drawn from outside of graph theory can be used to gain insight into scheduling problems.

REVIEW VOCABULARY

Algorithm A step-by-step description of how to solve a problem. (p. 34)

Brute force method The method that solves the traveling salesman problem (TSP) by enumerating all the Hamiltonian circuits and then selecting the one with minimum cost. (p. 37)

Complete graph A graph in which every pair of vertices is joined by an edge. (p. 36)

Critical path The longest path in an order-requirement digraph. The length of this path gives the earliest completion time for all the tasks making up the job consisting of the tasks in the digraph. (p. 49)

Fundamental principle of counting A method for counting outcomes of multistage processes. (p. 36)

Greedy algorithm An approach for solving an optimization problem, where at each stage of the algorithm the best (or cheapest) action is taken. Unfortunately, greedy algorithms do not always lead to optimal solutions. (p. 39)

Hamiltonian circuit A circuit using distinct edges of a graph that starts and ends at a particular vertex of the graph and visits each vertex once and only once. A Hamiltonian circuit can start at any one of its vertices. (p. 31)

Heuristic algorithm A method of solving an optimization problem that is "fast" but does not guarantee an optimal answer to the problem. (p. 42)

Kruskal's algorithm An algorithm developed by Joseph Kruskal (AT&T Research) that solves the minimum-cost spanning-tree problem by selecting edges in order of increasing cost, but in such a way that no edge forms a circuit with edges chosen earlier. It can be proved that this algorithm always produces an optimal solution. (p. 43)

Method of trees A visual method of carrying out the fundamental principle of counting. (p. 34)

Minimum-cost Hamiltonian circuit A Hamiltonian circuit in a graph with weights on the edges, for which the sum of the weights of the edges of the Hamiltonian circuit is as small as possible. (p. 34)

Minimum-cost spanning tree A spanning tree of a weighted connected graph having minimum cost. The cost of a tree is the sum of the weights on the edges of the tree. (p. 44)

Nearest-neighbor algorithm An algorithm for attempting to solve the TSP that begins at a "home" vertex and visits next that vertex not already visited that can be reached most cheaply. When all other vertices have been visited, the tour returns to home. This method may not give an optimal answer. (p. 39)

NP-complete problems A collection of problems, which includes the TSP, that appear to be very hard to solve quickly for an optimal solution. (p. 41)

Order-requirement digraph A directed graph that shows which tasks precede other tasks among the collection of tasks making up a job. (p. 48)

Sorted-edges algorithm An algorithm for attempting to solve the TSP where the edges added to the circuit being built up are selected in order of increasing cost, but no edge is chosen that would prevent a Hamiltonian circuit from forming. These edges must all be connected at the end, but not necessarily at earlier stages. The tour obtained may not have the lowest possible cost. (p. 40)

Spanning tree A subgraph of a connected graph that is a tree and includes all the vertices of the original graph. (p. 44)

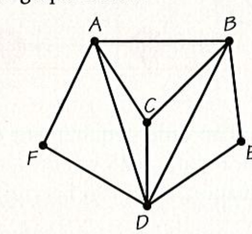
Traveling salesman problem (TSP) The problem of finding a minimum-cost Hamiltonian circuit in a complete graph where each edge has been assigned a cost (or weight). (p. 38)

Tree A connected graph with no circuits. (p. 34)

Weight A number assigned to an edge of a graph that can be thought of as a cost, distance, or time associated with that edge. (p. 34)

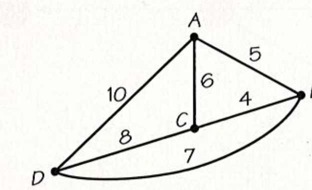
SKILLS CHECK

1. Which of the following describes a Hamiltonian circuit for the graph below?



(a) *ABCDFA*
(b) *AFDCBE*
(c) *ACBEDFA*
(d) *ACEBDFA*

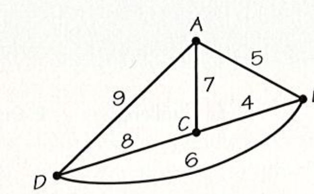
2. The cost of the nearest-neighbor tour (Hamiltonian circuit) that starts at vertex *A* for the graph below is ____________.



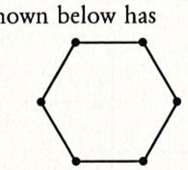
3. Suppose that after a hurricane, a van is dispatched to pick up five nurses at their homes and bring them to work at the local hospital. Which of these techniques is most likely to be useful in solving this problem?

(a) Finding an Euler circuit in a graph
(b) Solving a TSP (traveling salesman problem)
(c) Finding a minimum-cost spanning tree in a graph

4. The cost of the sorted-edges tour (Hamiltonian circuit) for the graph below is ____________.

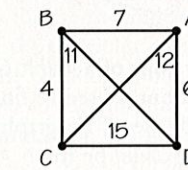


5. The graph shown below has



(a) no Hamiltonian circuit and no Euler circuit.
(b) an Euler circuit and a Hamiltonian circuit.
(c) no Hamiltonian circuit, but it has an Euler circuit.

6. The cost of the nearest-neighbor traveling salesman tour that starts at *B* for the following graph is ______.



7. When the sorted-edges method and nearest-neighbor method are applied to a complete graph on seven vertices with nonnegative weights,

(a) both methods always give the same optimal answer.
(b) both methods always give the same answer but that answer may not be optimal.
(c) neither method may give an optimal answer.

8. If a graph has *E* edges and *V* vertices as well as a Hamiltonian circuit, then the number of edges in the Hamiltonian circuit is ____________.

9. Paul has packed four ties, three shirts, and two pairs of pants for a trip. How many different outfits can he create if he never wears a tie?

(a) Fewer than 10
(b) Between 10 and 25
(c) More than 25

10. The number of different lunches that Jules can design by selecting one of three meats, one of three salads, and one of six vegetables is exactly ____________.

11. An ice-cream shop offers 3 types of cones, 20 flavors, and 4 different toppings (crushed peanuts, crushed almonds, chocolate bits, or corn flakes). If a customer is allergic to nuts, how many different choices can she choose from?

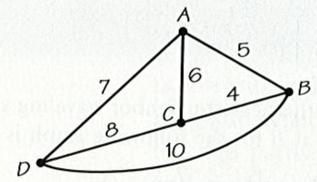
(a) 240
(b) 120
(c) 25

12. If a three-character password system must begin with a lowercase letter of the English alphabet followed by two decimal digits that may be repeated, the number of different possible passwords is ____________.

13. Assuming a graph with *E* edges and *V* vertices has a minimum-cost spanning tree *T*, which of the following statements *must* be true?

(a) The tree *T* has exactly *V* edges.
(b) The tree *T* includes every minimum-cost edge.
(c) The graph is connected.

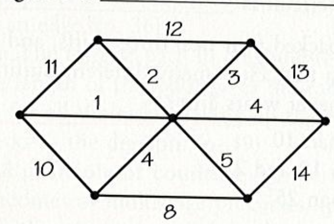
14. When arranged in increasing order, the weights of the edges in the following graph that are not part of the minimum-cost spanning tree selected when Kruskal's algorithm is applied are ______, ______, ______.



15. Assume that every edge of a graph G has a different cost. If Kruskal's algorithm is used to find the minimum-cost spanning tree T for graph G, which of the following statements *must* be true?

(a) Any other spanning tree for graph G will have more edges than T.
(b) Any other spanning tree for graph G will have a greater cost than T.
(c) The edge of graph G having greatest weight is included in T.

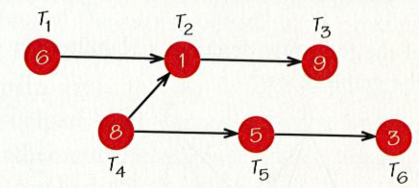
16. The smallest positive integer valued weight that x can have in the graph below so that it could not be selected by Kruskal's algorithm as an edge of a minimum-cost spanning tree is ______________.



17. If a graph contains a circuit, which of the following statements is true?

(a) The graph cannot be a tree.
(b) The graph must have the same number of vertices as edges.
(c) The graph is not connected.

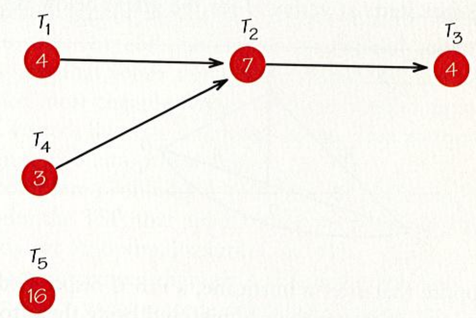
18. The earliest completion time (in minutes) for a job with the following order-requirement digraph is ______ .



19. Assume a job has an order-requirement digraph with five tasks whose critical path is 25 minutes in length. Based on this information, what can be said about the tasks?

(a) Each task takes exactly 5 minutes.
(b) Some task takes 25 minutes.
(c) The five tasks in total take at least 25 minutes.

20. The length of the critical path in the order-requirement digraph below is ______________ minutes.

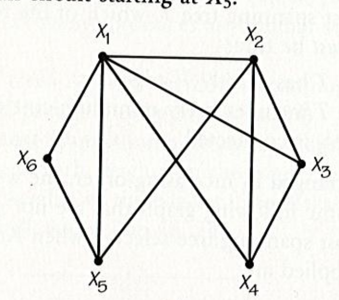


CHAPTER 2 EXERCISES

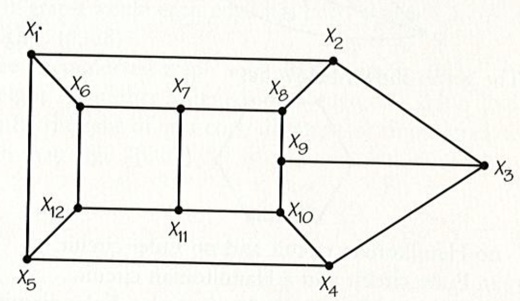
■ Challenge ◆ Discussion

2.1 Hamiltonian Circuits

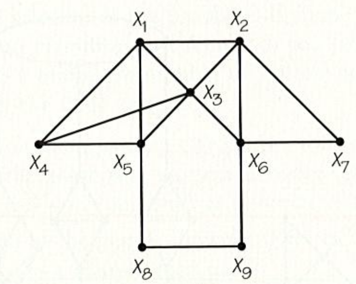
1. For the accompanying graphs (a) through (c), write a Hamiltonian circuit starting at X_5.



(*a*)

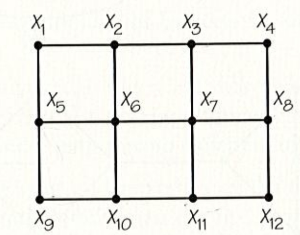


(*b*)

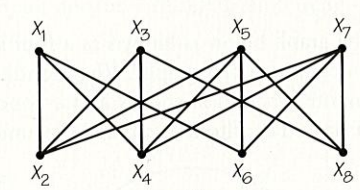


(*c*)

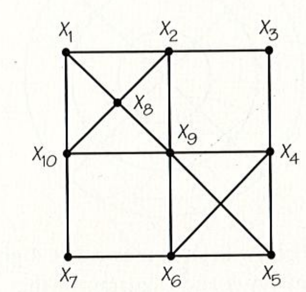
2. For the accompanying graphs (a) through (c) write down a Hamiltonian circuit starting at X_5.



(*a*)



(*b*)



(*c*)

3. If the edge X_2X_3 is erased from each of the graphs in Exercise 1, does the resulting graph still have a Hamiltonian circuit?

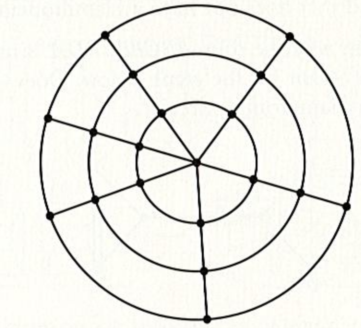
4. (a) If the vertex X_6 and the edges attached to X_6 are removed from the graphs in Exercise 1, do the new graphs that result still have Hamiltonian circuits?
(b) If you think of the graphs in Exercise 1 as communications networks, what interpretation might be given to the "removal" of a vertex and the edges attached as described in part (a)?

5. (a) If the edge X_6X_7 is removed (erased) from each of the graphs in Exercise 2, do the new graphs that result still have Hamiltonian circuits?
(b) If you think of the graphs in Exercise 2 as communications networks, what interpretation might be given to the "removal" of an edge as described in part (a)?

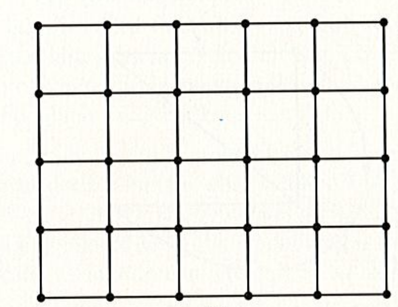
6. (a) Give examples of real-world situations that can be modeled using a graph and for which finding a Hamiltonian circuit in the graph would be of interest.
(b) For each of the examples you mention in part (a), can you adapt the question about the real-world situation involved so that finding an Eulerian circuit in the same graph would be of interest?

7. Suppose two Hamiltonian circuits are considered different if the collections of edges that they use are different. How many other Hamiltonian circuits can you find in the graph in Figure 2.1 that are different from the two discussed?

8. For each of the following graphs, add wiggly edges to indicate a Hamiltonian circuit.

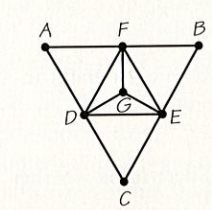


(*a*)

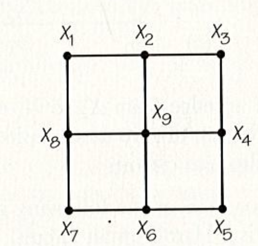


(*b*)

9. (a) Neither of the following graphs has a Hamiltonian circuit. Is it possible to add a single new edge to these graphs to obtain a new graph that has a Hamiltonian circuit?



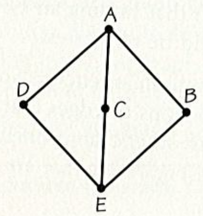
(*a*)



(*b*)

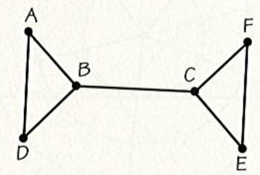
(b) Find an example of a graph that has no Hamiltonian circuit and will still have no Hamiltonian circuit no matter what single edge is added to it.
(c) Show that it is possible to add 4 additional edges to the graph diagram in part (b) above so that the resulting new graph will still have no Hamiltonian circuit.

10. Explain why the graph below has no Hamiltonian circuit.

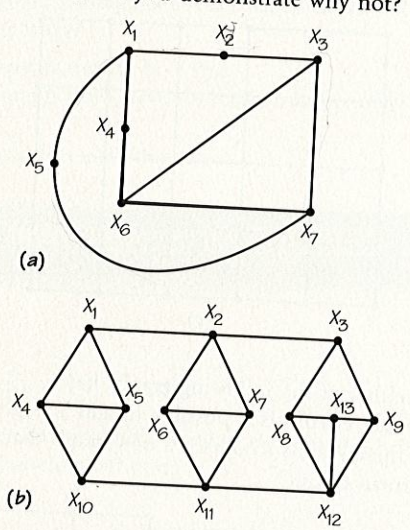


11. Use the graph shown in Exercise 10 to help you construct a connected graph for which every vertex has valence 3 and that does not have a Hamiltonian circuit.

■ **12.** Explain why the tour *ABCFECBDA* is not a Hamiltonian circuit for the graph below. Does this graph have a Hamiltonian circuit?

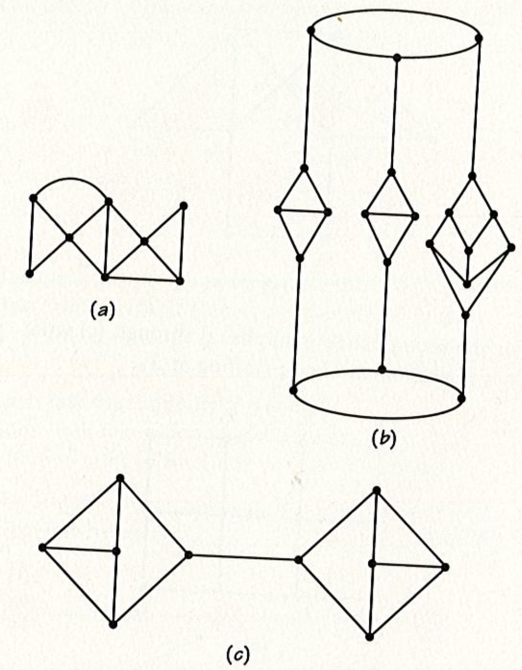


■ **13.** Do the following graphs have Hamiltonian circuits? If not, can you demonstrate why not?

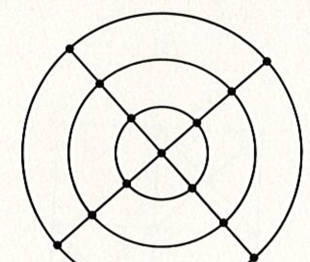


14. If an edge from X_2 to X_5 is added to each graph in Exercise 13, do the new graphs that result have a Hamiltonian circuit?

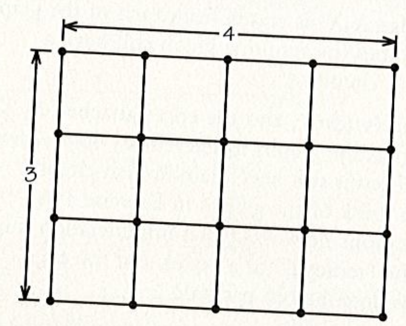
15. For each of the following graphs, determine whether there is a Hamiltonian circuit.



■ **16.** **(a)** The graph below is known as a four spokes and three concentric circles graph. What conditions on m and n guarantee that an m spokes and n concentric circles graph has a Hamiltonian circuit? (Assume $m \geq 2$, $n \geq 1$.)



(b) The graph below is known as a 3×4 grid graph. What conditions on m and n guarantee that an $m \times n$ grid graph has a Hamiltonian circuit?



Can you think of a real-world situation in which finding a Hamiltonian circuit in an $m \times n$ grid graph would represent a solution to the problem? If an $m \times n$ grid graph has no Hamiltonian circuit, can you find a tour that repeats a minimum number of vertices and starts and ends at the same vertex?

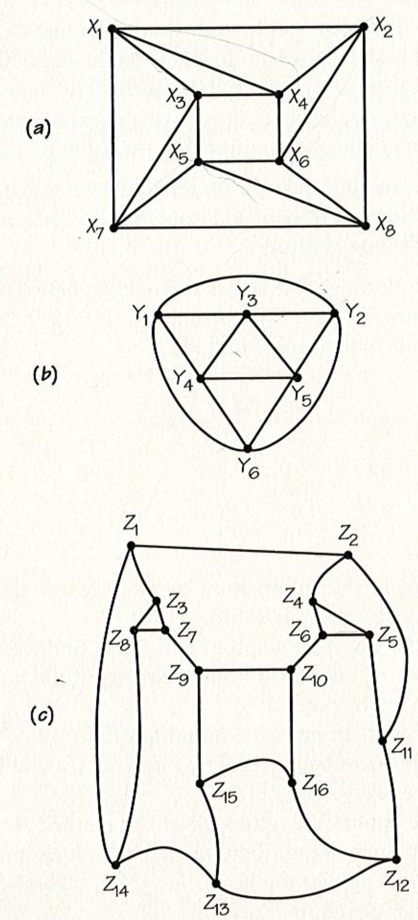
17. A Hamiltonian path in a graph is a tour of the vertices of the graph that visits each vertex once and only once and starts and ends at different vertices.

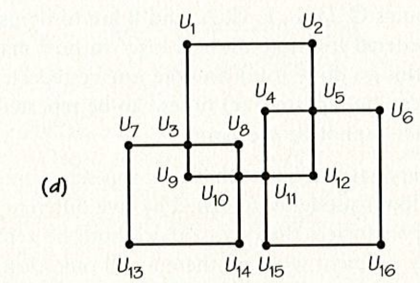
(a) For each of the graphs shown in Exercise 13, does the graph have a Hamiltonian path?
(b) Does each of these graphs have a Hamiltonian path that starts at X_1 and ends at X_2?
(c) Describe three real-world situations where finding a Hamiltonian path in a graph would be required.

18. Using the terminology of Exercise 17, draw a graph that has

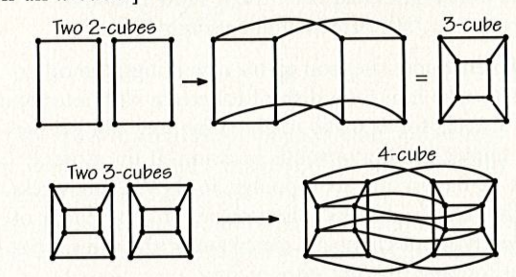
(a) a Hamiltonian path but no Hamiltonian circuit.
(b) an Euler circuit but no Hamiltonian path.
(c) a Hamiltonian path but no Euler circuit.

19. To practice your understanding of the concepts of Euler circuits and Hamiltonian circuits, determine for the following graphs (a) through (d) whether there is an Euler circuit and/or a Hamiltonian circuit. If so, write it down.





■ **20.** **(a)** The n-dimensional cube is obtained from two copies of an $(n-1)$-dimensional cube by joining corresponding vertices. (The process is illustrated for the 3-cube and the 4-cube in the following figure.) Can you show that every n-cube has a Hamiltonian circuit? [*Hint:* Show that if you know how to find a Hamiltonian circuit on an $(n-1)$-cube, then you can use two copies of this to build a Hamiltonian circuit on an n-cube.]



(b) Find formulas for the number of vertices and the number of edges of an n-cube.

21. If an edge is added from the vertex with subscript 4 to the vertex with subscript 5 in each graph in Exercise 19, which of the resulting graphs will have Hamiltonian circuits and which will have Euler circuits?

22. Find a family of graphs none of which have Hamiltonian circuits but for which adding a single edge to the first graph in the family creates a Hamiltonian circuit, adding two edges to the second graph in the family creates a Hamiltonian circuit, and so forth.

23. A Hamiltonian path in a graph is a tour of the vertices that visits each vertex once and only once and that starts and ends at different vertices.

(a) Draw an example of a graph that has no Hamiltonian path and where all the vertices are 3-valent.
(b) Draw a graph that has no Hamiltonian path but that does have an Euler circuit.
(c) By analogy with the Hamiltonian path, develop a definition of "Euler path."

24. **(a)** When going outside on a cold winter day, Jill can choose from three winter coats, five wool scarves, four pairs of boots, and three ski hats. How many outfits might her friends see her in?
(b) If Jill always insists on wearing her green wool scarf, how many outfits might her friends see her in?

25. The notes C, D, E, F, G, A, and B are to be used to form an ordered five-note musical logo. In how many ways can this be done if **(a)** no note can be repeated; **(b)** notes can be repeated; **(c)** notes can be repeated but all the notes cannot be the same?

26. A lottery game requires that a person select an upper- or lowercase letter followed by five different two-digit numbers (where the digits cannot both be zero). How many different ways are there to fill out a lottery ticket?

27. **(a)** In designing a security system for its accounts, a bank asks each customer to choose a five-digit number, all the digits to be distinct and nonzero. How many choices can a customer make?
(b) A suitcase with a liquid-crystal display allows one to unlock it with a specific combination of three capital letters that are not necessarily different. How many choices would a thief have to go through to be sure that all the possibilities had been tried? How does this compare to a "standard" combination lock?

28. To encourage her son to try new things, a mother offers to take him for a dish of ice cream with a topping once a week, for as many weeks as he does not get the same choice as on a previous occasion. If the store offers 12 flavors and six toppings, for how many weeks will she have to do this if her son never picks either of the two types of chocolate ice cream or the three types of nut topping that the store carries?

29. A large corporation has found that it has "outgrown" its current code system for routing interoffice mail. The current system places a code of three ordered, distinct nonzero digits on the mail. The new proposal calls for the use of two ordered capital letters. Does the new system have more code numbers than the old system? If so, how many more locations will the new system enable the company to encode over the current system?

30. Repeat Exercise 25a, except that exactly one of the notes in the musical logo must be a sharp and the note chosen to be sharped cannot appear elsewhere (for example, BCD#AG, where D# denotes D sharp).

◆ **31.** **(a)** In New York State, one type of license plate has three letters followed by three numbers. Suppose the digits from 0, 1, . . . , 9 can be used, except that all three digits cannot be zero, and that any letter from A to Z (repeats allowed) can be used. How many plates are possible?
(b) Investigate what schemes for license plates are used in your state and determine how many different plates are possible.

32. A restaurant offers 5 soups, 10 entrees, and 8 desserts. How many different choices for a meal can a customer make if one selection is made from each category? If 3 of the desserts are pies and the customer will never order pie, how many meals can the customer choose?

33. In the last several years, heavily populated regions that previously had only one area code have been divided into service areas with more than one area code. What is the largest number of different phone numbers that can be served using one area code? If an area code cannot begin with a zero, how many different area codes are possible?

34. **(a)** A credit-card company makes it easier for customers to memorize their PIN (personal identification number) by using a four-digit PIN that consists of three different digits selected from 0, 1, 2, . . . , 9 where one of the digits must be a zero, another is a nonzero digit that is repeated, and another is a digit different from these two. How many different PINs of this kind are there?
(b) How many PINs are possible if there are no restrictions on repeats of the 10 possible digits that can be used?

2.2 Traveling Salesman Problem

2.3 Helping Traveling Salesmen

35. Draw complete graphs with four, five, and six vertices. How many edges do these graphs have? Can you generalize to n vertices? How many TSP tours would these graphs have? (Tours yielding the same Hamiltonian circuit are considered the same.)

36. Calculate the values of 5!, 6!, 7!, 8!, 9!, and 10!. Then find the number of TSP tours in the complete graph with nine vertices.

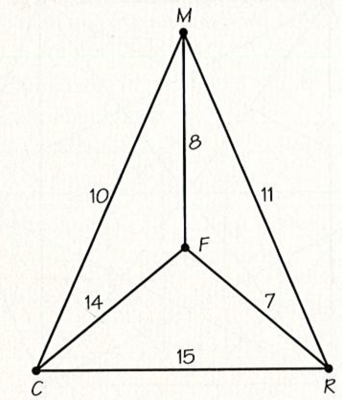
37. The following table shows the mileage between four cities: Springfield, Ill. (S); Urbana, Ill. (U); Effingham, Ill. (E); and Indianapolis, Ind. (I).

	E	I	S	U
E	–	147	92	79
I	147	–	190	119
S	92	190	–	88
U	79	119	88	–

(a) Represent this information by drawing a weighted complete graph on four vertices.
(b) Use the weighted graph in part (a) to find the cost of the three distinct Hamiltonian circuits in the graph. (List them starting at U.)
(c) Which circuit gives the minimum cost?
(d) Would there be any different in parts (b) and (c) if the start vertex were at I?
(e) If one applies the nearest-neighbor method starting at U, what circuit would be obtained? Does the answer change if one applies the nearest-neighbor algorithm starting at S? At E? At I?

(f) If one applies the sorted-edges method, what circuit would be obtained? Does one get the optimal answer?

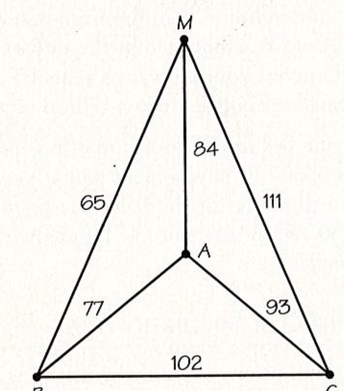
38. After a party at her house, Francine (F) has agreed to drive Mary (M), Rachel (R), and Constance (C) home. If the times (in minutes) to drive between her friends' homes are shown below, what route gets Francine back home the quickest?



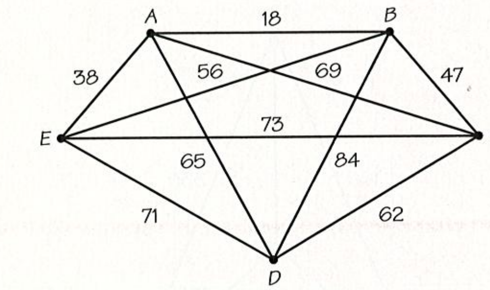
39. In Exercise 38, what route would Francine have to follow to get home as quickly as possible, assuming she promised to drive Mary home first?

40. In Exercise 38, Francine is planning to deliver her friends home and then spend the night at Rachel's house. What would her fastest route be?

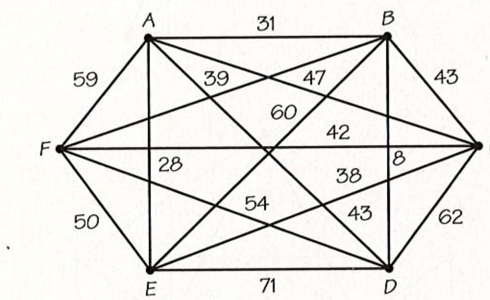
41. Starting from the location where she moors her boat (M), a fisherwoman wishes to visit three areas–A, B, and C–where she has set fishing nets. If the times (in minutes) between the locales are given in the figure below, what route to visit the three sites and return to the mooring place would be optimal?



42. **(a)** For the two complete graphs that follow, find the costs of the nearest-neighbor tour starting at B and of the tour generated by the sorted-edges algorithm.



(a)

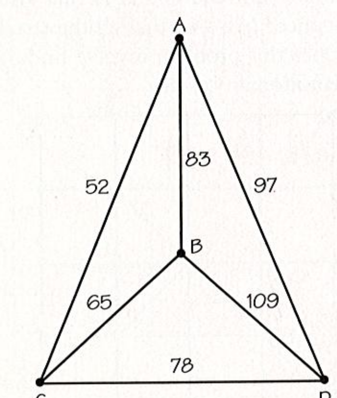


(b)

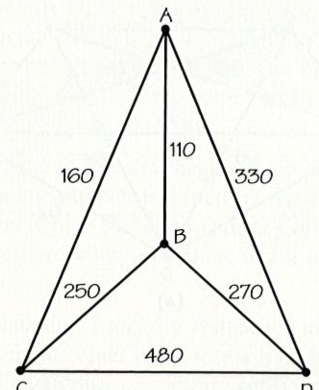
■ **(b)** How many Hamiltonian circuits would have to be examined to find a shortest route for part (a) by the brute force method?
(c) Invent an algorithm different from the sorted-edges and nearest-neighbor algorithms that is easy to apply for finding TSP solutions.

43. An airport limo must take its five passengers from the airport to different downtown hotels. Is this a traveling salesman problem, a Chinese postman problem, or an Euler circuit problem?

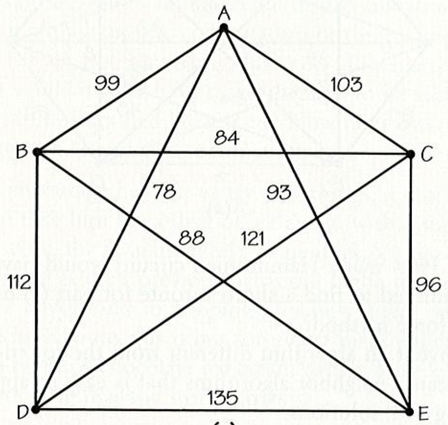
44. For each of the following graphs with weights, apply the nearest-neighbor method (starting at vertex A) and the sorted-edges method to find (it is hoped) a cheap tour.



(a)

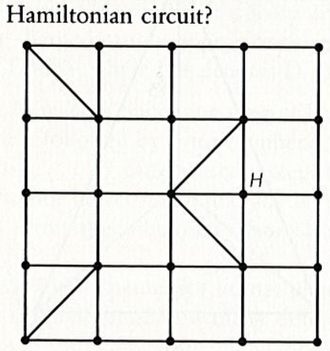


(b)



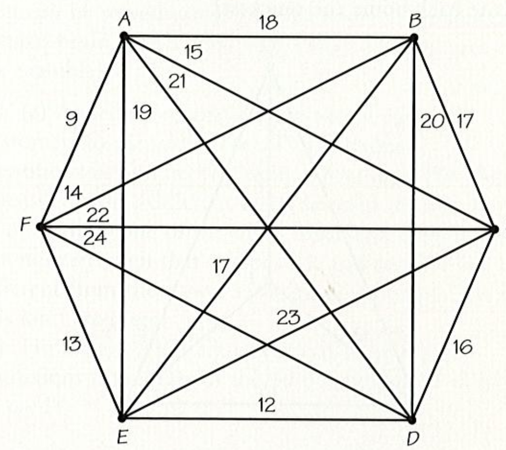
(c)

45. The following figure represents a town where there is a sewer located at each corner (where two or more streets meet). After every thunderstorm, the department of public works wishes to have a truck start at its headquarters (at vertex H) and make an inspection of sewer drains to be sure that leaves are not clogging them. Can a route start and end at H that visits each corner exactly once? (Assume that all the streets are two-way streets.) Does this problem involve finding an Euler circuit or a Hamiltonian circuit?



Assume that at equally spaced intervals along the blocks in this graph there are storm sewers that must be inspected after each thunderstorm to see if they are clogged. Is this a Hamiltonian circuit problem, an Euler circuit problem, or a Chinese postman problem? Find an optimal tour to do this inspection.

46. **(a)** Solve the six-city TSP shown in the diagram using the nearest-neighbor algorithm starting at vertex A and starting at vertex B.
(b) Apply the sorted-edges method.



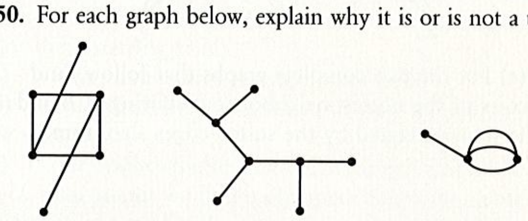
■ **47.** Construct an example of a complete graph of five vertices, with distinct weights on the edges for which the nearest-neighbor algorithm starting at a particular vertex and the sorted-edges algorithm yield different solutions for the traveling salesman problem. Can you find a five-vertex complete graph with weights on the edges in which the optimal solution, the nearest-neighbor solution, and the sorted-edges algorithm solution are all different?

■ **48.** If the brute force method of solving a 20-city TSP is employed, use a calculator to determine how many Hamiltonian circuits must be examined. How long would it take to determine the minimum-cost tour if the cost of tours could be computed at the rate of 1 billion per second? (Convert your answer to years by seeing how many years are equivalent to a billion seconds!)

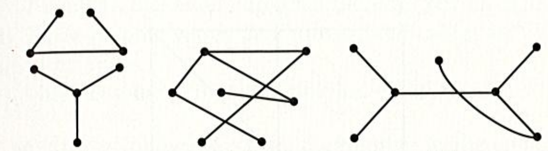
49. Suppose one has found an optimal tour for a given 10-city TSP problem to have weight 4200. Now suppose the weights on the edges of the complete graph are increased by 50. What can you say about the optimal tour and its weight?

2.4 Minimum-Cost Spanning Trees

50. For each graph below, explain why it is or is not a tree.

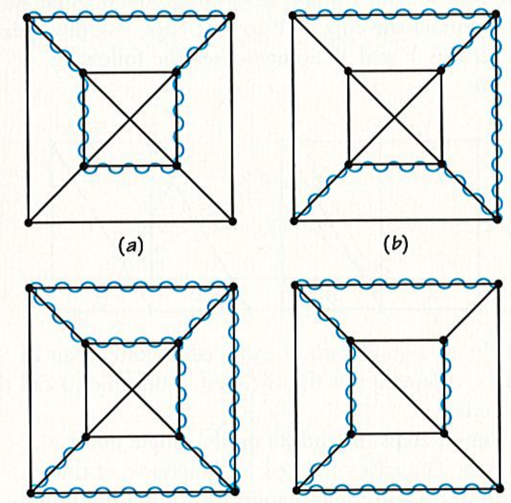


(a) (b) (c)



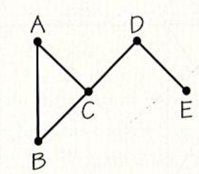
(d) (e) (f)

51. For each of the diagrams below, explain why the wiggly edges are not
(a) a spanning tree.
(b) a Hamiltonian circuit.

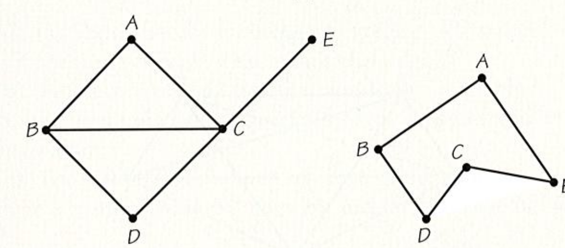


(a) (b) (c) (d)

52. Find all the spanning trees in the graphs below.

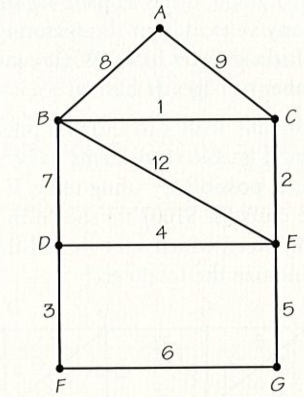


(a)

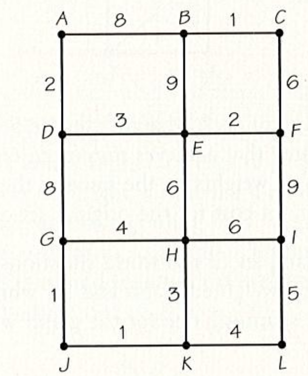


(b) (c)

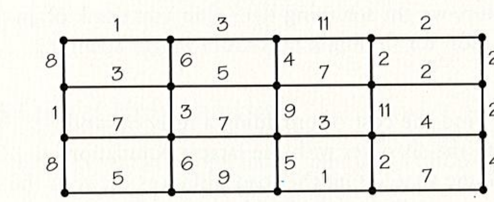
53. Use Kruskal's algorithm to find a minimum-cost spanning tree for the following graphs (a), (b), (c), and (d). In each case, what is the cost associated with the tree?



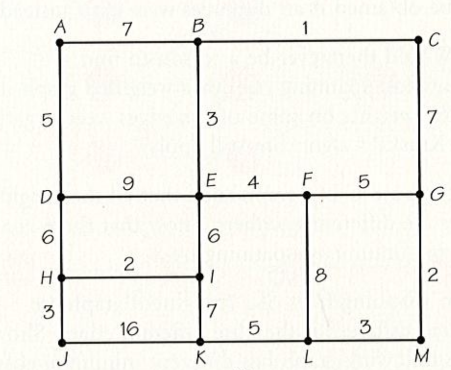
(a)



(b)



(c)

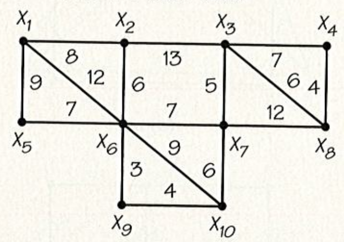


(d)

54. A connected graph G has 16 vertices. How many edges does a spanning tree of G have? How many vertices does a spanning tree of G have? What can one say about the number of edges G has?

55. A connected graph H has a spanning tree with 26 edges. How many vertices does the spanning tree have? How many vertices does H have? What can one say about the number of edges H has?

56. A large company wishes to install a pneumatic tube system that would enable small items to be sent between any of 10 locales, possibly by using relay. If the nonprohibitive costs (in \$100) are shown in the graph model below, between which sites should the tube be installed to minimize the total cost?



57. If the weight of each edge in Exercise 56 is increased by 3, will the tree that achieves minimum cost for the new collection of weights be the same as the one that achieves minimum cost for the original set of weights?

◆ **58.** Give examples of real-world situations that can be modeled using a weighted graph and for which finding a minimum-cost spanning tree for the graph would be of interest.

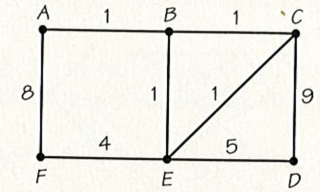
■ **59.** Can Kruskal's algorithm be modified to find a maximum-weight spanning tree? Can you think of an application for finding a maximum-weight spanning tree?

◆ **60.** Find the cost of providing a relay network between the six cities with the largest populations in your home state, using the road distances between the cities as costs. Does it follow that the same solution would be obtained if air distances were used instead?

■ **61.** Would there ever be a reason to find a minimum-cost spanning tree for a weighted graph in which the weights on some of the edges were negative? Would Kruskal's algorithm still apply?

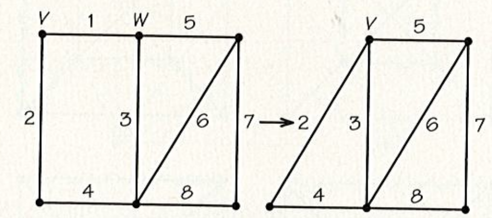
■ **62.** Suppose G is a graph such that all the weights on its edges are different numbers. Show that there is a unique minimum-cost spanning tree.

63. Two spanning trees of a (weighted) graph are considered different if they use different edges. Show that the following graph has different minimum-cost spanning trees, though all these different trees have the same cost.

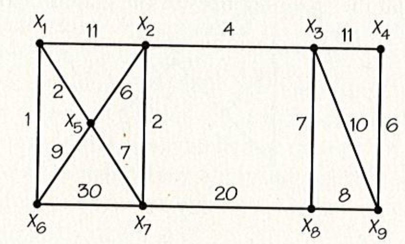


64. Let G be a graph with weights assigned to each edge. Consider the following algorithm:

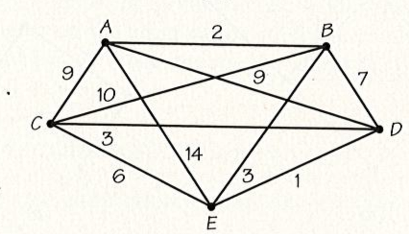
(a) Pick any vertex V of G.
(b) Select an edge E with a vertex at V that has a minimum weight. Let the other endpoints of E be W.
(c) Contract the edge VW so that edge VW disappears and vertices V and W coincide (see the following figures).



If in the new graph two or more edges join a pair of vertices, delete all but the cheapest. Continue to call the new vertex V.
(d) Repeat steps (b) and (c) until a single point is obtained. The edges selected in the course of this algorithm (called Prim's algorithm) form a minimum-cost spanning tree. Apply this algorithm to the following graphs.



(a)



(b)

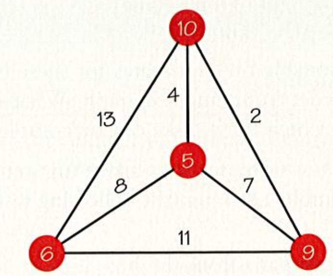
65. Determine whether each of the following statements is true or false for a minimum-cost spanning tree T for a weighted connected graph G:

(a) T contains a cheapest edge in the graph.
(b) T cannot contain a most expensive edge in the graph.

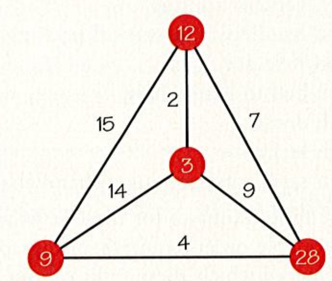
(c) T contains one fewer edge than there are vertices in G.
(d) There is some vertex in T to which all others are joined by edges.
(e) There is some vertex in T that has valence 3.

■ **66.** In the following graphs, the number in the circle for each vertex is the cost of installing equipment at the vertex if relaying must be done at the vertex, while the number on an edge indicates the cost of providing service between the endpoints of the edge.

In each case, find the minimum cost (allowing relays) for sending messages between any pair of vertices, taking vertex relay costs into account.



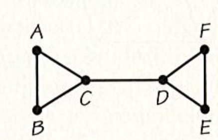
(a)



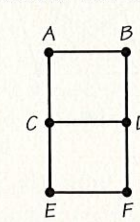
(b)

Would your answer be different if vertex relay costs were neglected? (*Warning:* Kruskal's algorithm cannot be used to answer the first question. This problem illustrates the value of having an algorithm over relying on "brute force.")

67. (a) Show that for each edge of graph J below there is a spanning tree of J that avoids that edge.
(b) For each spanning tree that you found in graph J, count the number of vertices and edges. Do you notice any pattern?
(c) For graph H below and each edge in the graph, is there a spanning tree that does not include that edge of H?



Graph H



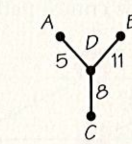
Graph J

68. (a) The table shown gives the "closeness" or distance values between four objects. Construct a four-vertex tree with weights on its edges such that the distances between pairs of vertices of the tree (as measured by the sum of the weights on the path in the tree between these vertices) give rise to this table.

	A	B	C	D
A	0	3	10	14
B	3	0	7	11
C	10	7	0	4
D	14	11	4	0

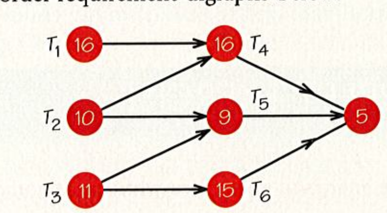
(b) Produce several real-world contexts that might give rise to the situation described here.

69. The figure below represents four objects using a tree with weights on the edges. Construct a table with four rows and four columns recording how "close" pairs of vertices in the tree are to each other. To find how close a pair of objects is, add together the weights along the path that joins these two objects.

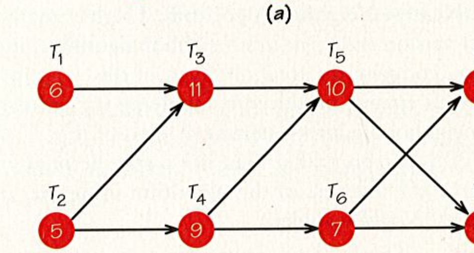


2.5 Critical-Path Analysis

70. Find the earliest completion time and critical paths for the order-requirement digraphs below.

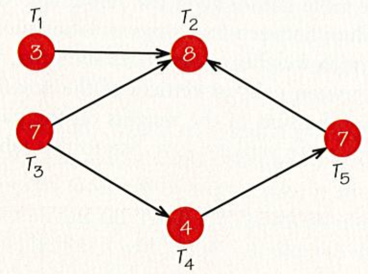


(a)

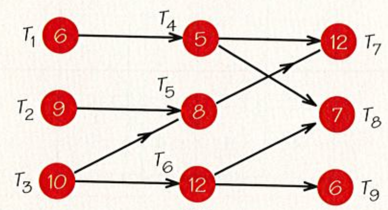


(b)

71. Find the earliest completion time and critical paths for the following order-requirement digraphs.



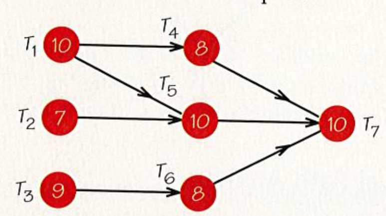
(*a*)



(*b*)

72. Construct an example of an order-requirement digraph with two different critical paths.

73. In the order-requirement digraph below, determine which tasks, if shortened, would reduce the earliest completion time and which would not. Then find the earliest completion time if task T_5 is reduced to time length 7. What is the new critical path?



74. For the order-requirement digraph in Exercise 73, find the critical path and the task(s) in the critical path whose time, when reduced the least, creates a new critical path.

75. To build a new addition on a house, the following tasks must be completed:

(a) Lay foundation.
(b) Erect sidewalls.
(c) Erect roof.
(d) Install plumbing.
(e) Install electric wiring.
(f) Lay tile flooring.
(g) Obtain building permits.
(h) Put in door that connects new room to existing house.
(i) Install track lighting on ceiling.
(j) Install wall air-conditioner.

Construct reasonable time estimates for these tasks and a reasonable order-requirement digraph. What is the fastest time in which these tasks can be completed?

76. At a large toy store, scooters arrive unassembled in boxes. To assemble a scooter, the following tasks must be performed:

TASK 1. Remove parts from the box.
TASK 2. Attach wheels to the footboard.
TASK 3. Attach vertical housing.
TASK 4. Attach handlebars to vertical housing.
TASK 5. Put on reflector tape.
TASK 6. Attach bell to handlebars.
TASK 7. Attach decals.
TASK 8. Attach kickstand.
TASK 9. Attach safety instructions to handlebars.

Give reasonable time estimates for these tasks and construct a reasonable order-requirement digraph. What is the earliest time by which these tasks can be completed?

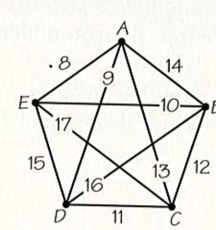
77. Construct an order-requirement digraph with six tasks that has three critical paths of length 26.

APPLET EXERCISES

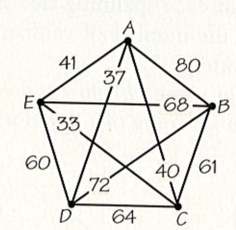
To do these exercises, go to www.whfreeman.com/fapp8e.

1. TSP: Nearest-Neighbor Algorithm. There is an extended version of the nearest-neighbor algorithm, in which you compare the total distances of the Hamiltonian circuits produced by applying the ordinary nearest-neighbor algorithm starting at each of the vertices of the graph (rather than just a specific one). Explore the effectiveness of this algorithm using the *TSP: Nearest-Neighbor* applet.

2. TSP: Sorted-Edges Algorithm. Go to the *TSP: Sorted Edges* applet, where you can apply the sorted-edges algorithm to see if it solves the traveling salesman problem for the following graphs (and others):

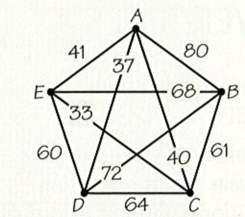


(*a*)

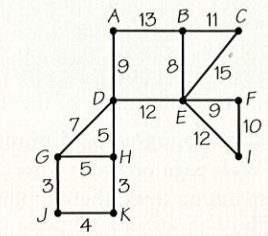


(*b*)

3. Kruskal's Algorithm. Go to the *Kruskal's Algorithm* applet, where you can apply Kruskal's algorithm to find the minimum-cost spanning trees in the following graphs (and others):



(*a*)



(*b*)

WRITING PROJECTS

1. Write an essay about a variety of situations in which you are personally involved for which a solution of the TSP is (perhaps implicitly) required. Explain under what circumstances it might be valuable to carry out a formal mathematical solution to such TSPs rather than use an ad hoc solution.

2. Construct an example, of the kind suggested on page 42, that shows that in a situation where three day campers must be picked up and brought to camp, it may make a difference if the optimization criterion is minimizing distance traveled by the camp bus versus minimizing average time that the children spend on the bus.

3. Determine the six largest cities in the state in which you live. By consulting a road atlas (or by some other means) construct the graph that represents the road distances between your hometown and these six other cities. Now apply (a) the nearest-neighbor method, (b) the sorted-edges method, and (c) the nearest neighbor from each city, and pick the minimum tour method to solve the associated TSP. Do you have reason to believe that the answers you get might include an optimum solution among them?

SUGGESTED READINGS

BODIN, LAWRENCE. Twenty years of routing and scheduling, *Operations Research,* 38 (1990): 571–579. A survey of real-world situations where routing and scheduling were used, written by a pioneer in this area.

DOLAN, ALAN, and JOAN ALDUS. *Networks and Algorithms: An Introductory Approach,* Wiley, Chichester, England, 1993. An excellent introduction to graph theory algorithms.

GUSFIELD, DAN. *Algorithms on Strings, Trees, and Sequences,* Cambridge University Press, New York, 1997. Details applications of graph theory in pattern recognition and reconstruction problems.

JONES, NEIL C., and PAVEL A. PEVZNER, *An Introduction to Bioinformatics Algorithms,* MIT Press, Cambridge, Mass., 2004. This book has material on how graph theory ideas, particularly those related to Hamiltonian circuits, are being used in molecular genetics and computational biology.

LAWLER, EUGENE, J. LENSTRA, RINNOY KAN, and D. SHMOYS, eds. *The Traveling Salesman Problem,* Prentice-Hall, Englewood Cliffs, N.J., 1985. Includes survey and technical articles on all aspects of the TSP.

LUCAS, WILLIAM, FRED ROBERTS, and ROBERT THRALL, eds. *Discrete and Systems Models,* vol. 3: *Modules in Applied Mathematics,* Springer-Verlag, New York, 1983. Chapter 6, "A Model for Municipal Street Sweeping Operations," by A. Tucker and L. Bodin, describes street-sweeping and related models in detail. Other chapters detail many recent applications of mathematics.

ROBERTS, FRED S., and BARRY TESMAN, *Applied Combinatorics,* 2nd ed., Pearson Prentice Hall, Upper Saddle River, N.J., 2004. The material on network-optimization problems is excellent.

ROBERTS, FRED. *Graph Theory and Its Applications to Problems of Society,* Society for Industrial and Applied Mathematics, Philadelphia, 1978. A very readable account of how graph theory is finding a wide variety of applications.